Weighted composition operators on the predual and dual Banach spaces of Beurling algebras on $\mathbb{Z}$

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Abstract

Let v be a weight sequence on $\mathbb{Z}$ and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. In this paper we study the boundedness, compactness and weak compactness of weighted composition operators $C_{\psi,\varphi}$ on predual Banach spaces $c_0(\mathbb{Z}, 1/v)$ and dual Banach spaces $\ell^\infty(\mathbb{Z}, 1/v)$ of Beurling algebras $\ell^1(\mathbb{Z}, v)$.

Key words: Weighted composition operator, Beurling algebra, compact operator, weakly compact operator
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1. Introduction

A *weight sequence* v on $\mathbb{Z}$ is a complex-valued function on $\mathbb{Z}$ such that $v(n) \geq 1$ for all $n \in \mathbb{Z}$ and $v(n+m) \leq v(n)v(m)$ for all $n, m \in \mathbb{Z}$. For example, for each $\alpha \geq 0$,

$$v_\alpha(n) = (1 + |n|)^\alpha \qquad (n \in \mathbb{Z})$$

defines a weight sequence on $\mathbb{Z}$.

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The *Beurling algebra* with weight sequence v on $\mathbb{Z}$ is the unital commutative Banach algebra

$$\ell^1(\mathbb{Z}, v) = \left\{ f : \mathbb{Z} \to \mathbb{C} \colon \sum_{n=-\infty}^{\infty} |f(n)|\, v(n) < \infty \right\}$$

with respect to the convolution product

$$(f \star g)(n) = \sum_{k=-\infty}^{\infty} f(n-k)g(k) \qquad (n \in \mathbb{Z})\,,$$

and the norm

$$\|f\|_v = \sum_{n=-\infty}^{\infty} |f(n)|\, v(n) \qquad \big(f \in \ell^1(\mathbb{Z}, v)\big)\,.$$

The dual Banach space of $\ell^1(\mathbb{Z}, v)$ is isometrically isomorphic to the space

$$\ell^\infty(\mathbb{Z}, 1/v) = \left\{ \lambda : \mathbb{Z} \to \mathbb{C} \colon \sup_{n \in \mathbb{Z}} \frac{|\lambda(n)|}{v(n)} < \infty \right\}$$

with the norm

$$\|\lambda\|_v = \sup_{n \in \mathbb{Z}} \frac{|\lambda(n)|}{v(n)} \qquad (\lambda \in \ell^\infty(\mathbb{Z}, 1/v))\,,$$

where the natural isometric isomorphism is the mapping $F \colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^1(\mathbb{Z}, v)^*$ defined by

$$F(\lambda)(f) = \sum_{n=-\infty}^{\infty} \lambda(n) f(n) \qquad \big(\lambda \in \ell^\infty(\mathbb{Z}, 1/v),\ f \in \ell^1(\mathbb{Z}, v)\big)\,.$$

A predual Banach space of $\ell^1(\mathbb{Z}, v)$ may be identified with the closed subspace

$$c_0(\mathbb{Z}, 1/v) = \left\{ \lambda \in \ell^\infty(\mathbb{Z}, 1/v) \colon \lim_{|n| \to \infty} \frac{|\lambda(n)|}{v(n)} = 0 \right\}$$

of $\ell^\infty(\mathbb{Z}, 1/v)$, by means of the isometric isomorphism $G \colon \ell^1(\mathbb{Z}, v) \to c_0(\mathbb{Z}, 1/v)^*$ given by

$$G(f)(\lambda) = \sum_{n=-\infty}^{\infty} \lambda(n) f(n) \qquad \big(f \in \ell^1(\mathbb{Z}, v),\ \lambda \in c_0(\mathbb{Z}, 1/v)\big)\,.$$

If $i \in \mathbb{Z}$ and e_i denotes the function on $\mathbb{Z}$ defined by $e_i(i) = 1$ and $e_i(n) = 0$ if $n \neq i$, note that $e_i \in c_0(\mathbb{Z}, 1/v)$ and $\|e_i\|_v = 1/v(i)$. In fact,

$$c_0(\mathbb{Z}, 1/v) = \overline{\mathrm{lin}}\, \{e_i \colon i \in \mathbb{Z}\}\,.$$

The book [4] by Dales contains a lot of information on these spaces.

A *weighted composition operator* $C_{\psi,\varphi}$ from $\ell^\infty(\mathbb{Z}, 1/v)$ into $\ell^\infty(\mathbb{Z}, 1/w)$ (of $c_0(\mathbb{Z}, 1/v)$ into $c_0(\mathbb{Z}, 1/w)$) is the linear operator defined by

$$C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi),$$

where ψ is a function from $\mathbb{Z}$ into $\mathbb{C}$ and φ a function from $\mathbb{Z}$ into $\mathbb{Z}$. The maps ψ and φ are referred as the *weight* and the *symbol* of the operator $C_{\psi,\varphi}$, respectively. Let $1_\mathbb{Z}$ be the function constantly 1 on $\mathbb{Z}$, and $I_\mathbb{Z}$ the identity function on $\mathbb{Z}$. The weighted composition operator $C_{\psi,\varphi}$ is called *multiplication operator* and denoted by M_ψ for the case $\varphi = I_\mathbb{Z}$, and it is known as *composition operator* and denoted by C_φ for the case $\psi = 1_\mathbb{Z}$. Note that $C_{\psi,\varphi} = M_\psi C_\varphi$.

The first paper dealing with boundedness and compactness of composition operators on spaces isomorphic to c_0 or l^∞ traces back to the work of Madigan and Matheson [7]. This work was later refined by Montes-Rodríguez [8, 9]. On the other hand, the boundedness, compactness and weak compactness of weighted composition operators defined on different weighted Banach spaces have been investigated, for example, in [1, 2, 3, 10, 12, 15, 16, 17, 18, 19, 20]. We refer the reader to the monograph by Singh and Manhas [14] and the references therein for a comprehensive treatment of this subject. In the context of weighted sequences spaces, we can cite, among others, the papers [5, 6, 11, 13].

Our aim in this paper is to study these properties for the weighted composition operators $C_{\psi,\varphi}$ on $\ell^\infty(\mathbb{Z}, 1/v)$-spaces and $c_0(\mathbb{Z}, 1/v)$-spaces by means of the function-theoretic properties of ψ and φ.

We now describe briefly the content of the paper. In Sections 2 and 3, we characterize the bounded operators and the compact operators $C_{\psi,\varphi}$ from $\ell^\infty(\mathbb{Z}, 1/v)$ into $\ell^\infty(\mathbb{Z}, 1/w)$, and also defined from $c_0(\mathbb{Z}, 1/v)$ into $c_0(\mathbb{Z}, 1/w)$, in terms of the quantity $|\psi(n)|\, v(\varphi(n))/w(n)$. Namely, the "big-oh" condition on this quantity describes the class of bounded operators, while that the corresponding "little-oh" condition determines the subclass of compact operators. In Section 4, we prove that every weakly compact operator $C_{\psi,\varphi}$ between $\ell^\infty(\mathbb{Z}, 1/v)$-spaces or $c_0(\mathbb{Z}, 1/v)$-spaces is compact.

2. Boundedness of $C_{\psi,\varphi}$

Let v be a weight sequence on $\mathbb{Z}$. An easy verification yields

$$v(n) = \max\left\{|\lambda(n)| : \lambda \in \ell^\infty(\mathbb{Z}, 1/v),\ \|\lambda\|_v \le 1\right\} \qquad (n \in \mathbb{Z})\,. \tag{1}$$

Note that $v \in \ell^\infty(\mathbb{Z}, 1/v)$ and $\|v\|_v = 1$.

For each $n \in \mathbb{Z}$, let $\delta_n \colon \ell^\infty(\mathbb{Z}, 1/v) \to \mathbb{C}$ be the linear functional defined by

$$\delta_n(\lambda) = \lambda(n) \qquad (\lambda \in \ell^\infty(\mathbb{Z}, 1/v)).$$

Observe that δ_n is bounded and, by (1),

$$\|\delta_n\|_v := \sup\{|\delta_n(\lambda)| : \lambda \in \ell^\infty(\mathbb{Z}, 1/v),\ \|\lambda\|_v \le 1\} = v(n). \tag{2}$$

Now, for each $n \in \mathbb{Z}$, let β_n be the restriction map of δ_n to $c_0(\mathbb{Z}, 1/v)$. Clearly, $\beta_n \in c_0(\mathbb{Z}, 1/v)^*$ and

$$\|\beta_n\|_v := \sup\{|\lambda(n)| : \lambda \in c_0(\mathbb{Z}, 1/v),\ \|\lambda\|_v \le 1\} \le v(n).$$

In fact, we have

$$\|\beta_n\|_v = v(n) \qquad (n \in \mathbb{Z}). \tag{3}$$

Indeed, fix $m \in \mathbb{Z}$ and define

$$\lambda_m(n) = \begin{cases} 1 & \text{if } n = 0, \\ v(m) & \text{if } n = m, \\ v(n)/n & \text{in other case.} \end{cases}$$

It is clear that $\lambda_m \in c_0(\mathbb{Z}, 1/v)$ with $\|\lambda_m\|_v \le 1$, and so $\|\beta_m\|_v \ge |\lambda_m(m)| = v(m)$.

On the other hand, it is said that a weight sequence v is *typical* if $v(n) \to \infty$ as $|n| \to \infty$.

We begin by characterizing the bounded weighted composition operators between $c_0(\mathbb{Z}, 1/v)$-spaces in the following form.

Theorem 2.1. *Let v, w be weight sequences on $\mathbb{Z}$ such that v is typical and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. Then the operator $C_{\psi,\varphi} \colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is bounded if and only if $\psi \in c_0(\mathbb{Z}, 1/w)$ and*

$$\sup_{n \in \mathbb{Z}} (|\psi(n)|\, v(\varphi(n))/w(n)) < \infty.$$

In this case, the norm of the operator $C_{\psi,\varphi}$ satisfies

$$\|C_{\psi,\varphi}\| = \sup_{n \in \mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}.$$

Proof. Assume that $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is bounded. If

$$\sup_{n\in\mathbb{Z}}\frac{|\psi(n)|\,v(\varphi(n))}{w(n)}=\infty,$$

we then can take a sequence $\{n_k\}_{k\in\mathbb{N}}\subset\mathbb{Z}$ satisfying

$$|\psi(n_k)|\,v(\varphi(n_k))>kw(n_k),\qquad \forall k\in\mathbb{N}.$$

Using (3), for each $k\in\mathbb{N}$ there exists $\lambda_{n_k}\in c_0(\mathbb{Z},1/v)$ with $\|\lambda_{n_k}\|_v\le 1$ such that

$$|\lambda_{n_k}(\varphi(n_k))|>\frac{v(\varphi(n_k))}{2}.$$

Now, since

$$\begin{aligned}\|C_{\psi,\varphi}(\lambda_{n_k})\|_w&=\sup_{n\in\mathbb{Z}}\frac{|\psi(n)\lambda_{n_k}(\varphi(n))|}{w(n)}\\&\ge\frac{|\psi(n_k)\lambda_{n_k}(\varphi(n_k))|}{w(n_k)}\\&>\frac{|\psi(n_k)|\,v(\varphi(n_k))}{2w(n_k)}>\frac{k}{2}\end{aligned}$$

for all $k\in\mathbb{N}$, we conclude that the operator $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is not bounded, a contradiction. Moreover, since v is typical, it is clear that $1_{\mathbb{Z}}\in c_0(\mathbb{Z},1/v)$ and so $\psi=C_{\psi,\varphi}(1_{\mathbb{Z}})\in c_0(\mathbb{Z},1/w)$.

Conversely, suppose that $\psi\in c_0(\mathbb{Z},1/w)$ and

$$M:=\sup_{n\in\mathbb{Z}}\frac{|\psi(n)|\,v(\varphi(n))}{w(n)}<\infty.$$

If we show that $C_{\psi,\varphi}$ maps $c_0(\mathbb{Z},1/v)$ into $c_0(\mathbb{Z},1/w)$, then $C_{\psi,\varphi}$ is bounded. For it, let $\lambda\in c_0(\mathbb{Z},1/v)$ and $\varepsilon>0$ be given. Then there exists $k_1>0$ such that

$$|n|\ge k_1\quad\Rightarrow\quad\frac{|\lambda(n)|}{v(n)}<\frac{\varepsilon}{1+M}.$$

Moreover, there exists $k_2>k_1$ such that

$$|n|\ge k_2\quad\Rightarrow\quad\frac{|\psi(n)|}{w(n)}<\frac{\varepsilon}{1+\sup_{|s|\le k_1}|\lambda(s)|}.$$

Let $n \in \mathbb{Z}$ be for which $|n| \geq k_2$. If $|\varphi(n)| > k_1$, we have

$$\frac{|C_{\psi,\varphi}(\lambda)(n)|}{w(n)} = \frac{|\psi(n)\lambda(\varphi(n))|}{w(n)} = \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} \frac{|\lambda(\varphi(n))|}{v(\varphi(n))} < M\frac{\varepsilon}{1+M} < \varepsilon,$$

and if $|\varphi(n)| \leq k_1$,

$$\frac{|C_{\psi,\varphi}(\lambda)(n)|}{w(n)} = \frac{|\psi(n)\lambda(\varphi(n))|}{w(n)} \leq \frac{|\psi(n)|}{w(n)} \sup_{|s|\leq k_1} |\lambda(s)| < \varepsilon.$$

Hence $C_{\psi,\varphi}(\lambda) \in c_0(\mathbb{Z}, 1/w)$ as required.

It remains to compute the norm of $C_{\psi,\varphi}$. From above we readily deduce

$$\|C_{\psi,\varphi}\| \leq \sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}.$$

For the converse, notice first that the adjoint operator $C^*_{\psi,\varphi}\colon c_0(\mathbb{Z}, 1/w)^* \to c_0(\mathbb{Z}, 1/v)^*$ satisfies

$$C^*_{\psi,\varphi}\left(\beta_n\right) = \psi(n)\beta_{\varphi(n)} \qquad (n \in \mathbb{Z})\,. \tag{4}$$

Indeed, we have

$$C^*_{\psi,\varphi}(\beta_n)(\lambda) = \beta_n\left(\psi \cdot (\lambda \circ \varphi)\right) = \psi(n)\lambda(\varphi(n)) = \psi(n)\beta_{\varphi(n)}(\lambda)$$

for all $\lambda \in c_0(\mathbb{Z}, 1/v)$. Using the equalities (4) and (3), we have

$$\|C_{\psi,\varphi}\| = \left\|C^*_{\psi,\varphi}\right\| \geq \frac{\left\|C^*_{\psi,\varphi}\left(\beta_n\right)\right\|_v}{\|\beta_n\|_w} = \frac{|\psi(n)|\left\|\beta_{\varphi(n)}\right\|_v}{\|\beta_n\|_w} = \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}$$

for all $n \in \mathbb{Z}$, and hence

$$\sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} \leq \|C_{\psi,\varphi}\|\,,$$

as we desired. □

Remark 2.2. *According to the proof of Theorem 2.1, the boundedness of the operator* $C_{\psi,\varphi}\colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ *implies that*

$$\sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} < \infty$$

without assuming that the weight sequence v *is typical.*

Our next aim is to get the analogous version of Theorem 2.1 for operators $C_{\psi,\varphi}$ on $\ell^\infty(\mathbb{Z}, 1/v)$-spaces. For it, we first identify this operator with its double adjoint with the aid of the natural identifications $F\colon \ell^\infty(\mathbb{Z}, 1/w) \to \ell^1(\mathbb{Z}, w)^*$ and $G\colon \ell^1(\mathbb{Z}, w) \to c_0(\mathbb{Z}, 1/w)^*$ defined in the first section.

Lemma 2.3. *Let v, w be weight sequences on $\mathbb{Z}$ and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. Then the operator $C_{\psi,\varphi}\colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/w)$ coincides with $\Phi\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}\Phi^{-1}$ where Φ is the isometric isomorphism $F^{-1} \circ G^*\colon c_0(\mathbb{Z}, 1/v)^{**} \to \ell^\infty(\mathbb{Z}, 1/v)$.*

Proof. If i denotes the inclusion map from $c_0(\mathbb{Z}, 1/v)$ into $\ell^\infty(\mathbb{Z}, 1/v)$, and J the canonical injection of $c_0(\mathbb{Z}, 1/v)$ into $c_0(\mathbb{Z}, 1/v)^{**}$, we have

$$[(G^* \circ J)(\lambda)](f) = [J(\lambda)](G(f)) = [G(f)](\lambda) = [F(\lambda)](f) = [(F \circ i)(\lambda)](f)$$

for all $\lambda \in c_0(\mathbb{Z}, 1/v)$ and $f \in \ell^1(\mathbb{Z}, v)$. Hence $F^{-1} \circ G^* \circ J = i$, and so

$$(F^{-1} \circ G^*)(J(\lambda))(n) = \lambda(n) = \beta_n(\lambda) = J(\lambda)(\beta_n) \qquad (\lambda \in c_0(\mathbb{Z}, 1/v),\ n \in \mathbb{Z}).$$

Using the w^*-denseness of $J(c_0(\mathbb{Z}, 1/v))$ in $c_0(\mathbb{Z}, 1/v)^{**}$ by Goldstine's Theorem, we obtain that

$$(F^{-1} \circ G^*)(x)(n) = x(\beta_n) \qquad (x \in c_0(\mathbb{Z}, 1/v)^{**},\ n \in \mathbb{Z}).$$

If we denote by Φ the isometric isomorphism $F^{-1} \circ G^*\colon c_0(\mathbb{Z}, 1/v)^{**} \to \ell^\infty(\mathbb{Z}, 1/v)$, the preceding equalities may be expressed in the form

$$\Phi(J(\lambda))(n) = \lambda(n) \qquad (\lambda \in c_0(\mathbb{Z}, 1/v),\ n \in \mathbb{Z}), \tag{5}$$

$$\Phi(x)(n) = x(\beta_n) \qquad (x \in c_0(\mathbb{Z}, 1/v)^{**},\ n \in \mathbb{Z}). \tag{6}$$

Now, given $\lambda \in \ell_\infty(\mathbb{Z}, 1/v)$ and $n \in \mathbb{Z}$, we conclude that

$$\begin{aligned}
\left(\Phi\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}\Phi^{-1}\right)(\lambda)(n) &= \Phi\left(\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}\Phi^{-1}(\lambda)\right)(n)\\
&= \left(\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}\Phi^{-1}(\lambda)\right)(\beta_n)\\
&= \left(\Phi^{-1}(\lambda)\circ\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{*}\right)(\beta_n)\\
&= \Phi^{-1}(\lambda)\left(\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{*}(\beta_n)\right)\\
&= \Phi^{-1}(\lambda)\left(\psi(n)\beta_{\varphi(n)}\right)\\
&= \psi(n)\Phi^{-1}(\lambda)\left(\beta_{\varphi(n)}\right)\\
&= \psi(n)\Phi\left(\Phi^{-1}(\lambda)\right)(\varphi(n))\\
&= \psi(n)\lambda(\varphi(n))\\
&= C_{\psi,\varphi}(\lambda)(n).
\end{aligned}$$

We have used the formula (6) in the second and seventh equalities, and (4) in the fifth one. $\square$

Theorem 2.4. *Let v, w be weight sequences on $\mathbb{Z}$ and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. Then the operator $C_{\psi,\varphi}$ from $\ell^\infty(\mathbb{Z}, 1/v)$ into $\ell^\infty(\mathbb{Z}, 1/w)$ is bounded if and only if*

$$\sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} < \infty.$$

In this case, the norm of the operator $C_{\psi,\varphi}$ is given by the expression

$$\|C_{\psi,\varphi}\| = \sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}.$$

Proof. Assume that $C_{\psi,\varphi}\colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/w)$ is bounded. Then, by Lemma 2.3, so also is $\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**} : c_0(\mathbb{Z}, 1/v)^{**} \to c_0(\mathbb{Z}, 1/w)^{**}$. It follows that $C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)} : c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is bounded. Consequently,

$$\sup_{n\in\mathbb{Z}} \left(|\psi(n)|\, v(\varphi(n))/w(n)\right) < \infty$$

by Theorem 2.1 and Remark 2.2.

Conversely, if $M := \sup_{n\in\mathbb{Z}} \left(|\psi(n)|\, v(\varphi(n))/w(n)\right) < \infty$, we have

$$|\psi(n)|\, v(\varphi(n)) \le M w(n)$$

for all $n \in \mathbb{Z}$. It follows that

$$\frac{|C_{\psi,\varphi}(\lambda)(n)|}{w(n)} = \frac{|\psi(n)\lambda(\varphi(n))|}{w(n)} = \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} \frac{|\lambda(\varphi(n))|}{v(\varphi(n))} \leq M \left\|\lambda\right\|_v$$

for all $\lambda \in \ell^\infty(\mathbb{Z}, 1/w)$ and $n \in \mathbb{Z}$, and thus $C_{\psi,\varphi} \colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/w)$ is bounded.

In order to calculate the norm $\|C_{\psi,\varphi}\|$, first notice that the preceding argument yields

$$\|C_{\psi,\varphi}\| \leq \sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}.$$

On the other hand, using that $C^*_{\psi,\varphi}(\delta_n) = \psi(n)\delta_{\varphi(n)}$ for all $n \in \mathbb{Z}$ and the equality (2), we infer that

$$\|C_{\psi,\varphi}\| = \left\|C^*_{\psi,\varphi}\right\| \geq \frac{\left\|C^*_{\psi,\varphi}(\delta_n)\right\|_v}{\left\|\delta_n\right\|_w} = \frac{|\psi(n)| \left\|\delta_{\varphi(n)}\right\|_v}{\left\|\delta_n\right\|_w} = \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}$$

for all $n \in \mathbb{Z}$, and so

$$\sup_{n\in\mathbb{Z}} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} \leq \|C_{\psi,\varphi}\|,$$

as we wanted. □

3. Compactness of $C_{\psi,\varphi}$

We start with the following lemma that characterizes the relatively compact subsets of $c_0(\mathbb{Z}, 1/v)$.

Lemma 3.1. *Let v be a weight sequence on $\mathbb{Z}$. A set K in $c_0(\mathbb{Z}, 1/v)$ is relatively compact if and only if it is bounded and satisfies*

$$\lim_{|n|\to\infty} \sup_{\lambda\in K} \frac{|\lambda(n)|}{v(n)} = 0. \tag{7}$$

Proof. Assume first that K is relatively compact and let $\varepsilon > 0$. Then there exist $\lambda_1, \ldots,$ $\lambda_m \in K$ such that for any $\lambda \in K$ there is some $1 \leq i \leq m$ such that $\|\lambda - \lambda_i\|_v < \varepsilon/2$. Moreover, since $\lambda_1, \ldots, \lambda_m \in c_0(\mathbb{Z}, 1/v)$, there is $k > 0$ such that $|\lambda_i(n)|/v(n) < \varepsilon/2$ if $|n| > k$, $1 \leq i \leq m$. If $\lambda \in K$, then $\|\lambda - \lambda_i\|_v < \varepsilon/2$ for some $1 \leq i \leq m$ and so

$$\frac{|\lambda(n)|}{v(n)} \leq \frac{|\lambda(n) - \lambda_i(n)|}{v(n)} + \frac{|\lambda_i(n)|}{v(n)} \leq \|\lambda - \lambda_i\|_v + \frac{|\lambda_i(n)|}{v(n)} < \varepsilon$$

whenever $|n| > k$. This proves (7).

Conversely, suppose that K is a bounded subset of $c_0(\mathbb{Z}, 1/v)$ which satisfies the condition (7). Since $c_0(\mathbb{Z}, 1/v)$ is a Banach space, it suffices to prove that K is totally bounded. Let $\varepsilon > 0$ be fixed. By (7) we can find $k > 0$ such that

$$\left\| \sum_{|i|>k}^{\infty} \lambda(i) e_i \right\|_v = \sup_{|n|>k} \frac{|\lambda(n)|}{v(n)} < \frac{\varepsilon}{4}$$

for all $\lambda \in K$. If $[k]$ stands for the integer part of k, consider the set

$$A_{[k]} = \left\{ \sum_{i=-[k]}^{[k]} \lambda(i) e_i \colon \lambda \in K \right\}.$$

Since K is bounded in $c_0(\mathbb{Z}, 1/v)$, there is a $M > 0$ such that $\|\lambda\|_v < M$ for all $\lambda \in K$, and thus

$$\left\| \sum_{i=-[k]}^{[k]} \lambda(i) e_i \right\|_v = \sup_{|n|\leq[k]} \frac{|\lambda(n)|}{v(n)} \leq \|\lambda\|_v < M$$

for all $\lambda \in K$. Hence $A_{[k]}$ is bounded in the finite-dimensional Banach space $\operatorname{lin}\{e_i \colon |i| \leq [k]\}$ and, in particular, it is totally bounded. Then there exists a finite set $F \subset K$ such that for any $\lambda \in K$ there is some $\gamma \in F$ for which $\left\| \sum_{i=-[k]}^{[k]} (\lambda(i) - \gamma(i))\, e_i \right\|_v < \varepsilon/2$, and so

$$\|\lambda - \gamma\|_v \leq \left\| \sum_{i=-[k]}^{[k]} (\lambda(i) - \gamma(i))\, e_i \right\|_v + \left\| \sum_{|i|>[k]}^{\infty} (\lambda(i) - \gamma(i))\, e_i \right\|_v < \varepsilon.$$

□

We now are ready to state the main result of this section.

Theorem 3.2. *Let v, w be weight sequences on $\mathbb{Z}$ such that v is typical and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. Then the operator $C_{\psi,\varphi} \colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is compact if and only if*

$$\lim_{|n|\to\infty} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} = 0.$$

Proof. Recall that a linear operator between Banach spaces $T\colon E \to F$ is compact if and only if $\{T(x)\colon \|x\| \le 1\}$ is relatively compact in F. Let $B = \{\lambda \in c_0(\mathbb{Z}, 1/v)\colon \|\lambda\|_v \le 1\}$. According to Lemma 3.1, the operator $C_{\psi,\varphi}\colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is compact if and only if the following are satisfied:

i) $C_{\psi,\varphi}(B)$ is bounded in $c_0(\mathbb{Z}, 1/w)$,
ii) $\lim_{|n|\to\infty} \sup_{\lambda\in B} \frac{|\psi(n)\lambda(\varphi(n))|}{w(n)} = 0$.

For each $n \in \mathbb{Z}$, by using (3) we have

$$\sup_{\lambda\in B} \frac{|\psi(n)\lambda(\varphi(n))|}{w(n)} = \sup_{\lambda\in B} \frac{|\psi(n)| \left|\beta_{\varphi(n)}(\lambda)\right|}{w(n)} = \frac{|\psi(n)|\, v(\varphi(n))}{w(n)}.$$

Note that ii) implies i). Indeed, if ii) is satisfied, we deduce that ψ is in $c_0(\mathbb{Z}, 1/w)$ since $1_{\mathbb{Z}} \in B$, and $\sup_{n\in\mathbb{Z}} \left(|\psi(n)|\, v(\varphi(n))/w(n)\right)$ is finite. Then i) holds by Theorem 2.1. Therefore it follows that $C_{\psi,\varphi}\colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is compact if and only if

$$\lim_{|n|\to\infty} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} = 0.$$

□

Using Lemma 2.3, the Schauder's Theorem and Theorem 3.2, we deduce the following

Corollary 3.3. *Let v, w be weight sequences on $\mathbb{Z}$ such that v is typical and let ψ, φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z}) \subset \mathbb{Z}$. Then the operator $C_{\psi,\varphi}\colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/w)$ is compact if and only if*

$$\lim_{|n|\to\infty} \frac{|\psi(n)|\, v(\varphi(n))}{w(n)} = 0.$$

Proof. Note that $C_{\psi,\varphi}\colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/w)$ is compact if and only if $\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)} \right)^{**}$ is compact from $c_0(\mathbb{Z}, 1/v)^{**}$ into $c_0(\mathbb{Z}, 1/w)^{**}$ (by Lemma 2.3) if and only if $C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/w)$ is compact (by Schauder's Theorem) if and only if $\lim_{|n|\to\infty} \left(|\psi(n)|\, v(\varphi(n))/w(n)\right) = 0$ (by Theorem 3.2). □

In the cases that $\varphi = I_{\mathbb{Z}}$ and $\psi = 1_{\mathbb{Z}}$, our results provide characterizations for the boundedness and compactness of multiplication operators M_ψ and composition operators C_φ, respectively. In particular, we point out the following results for compact operators on $\ell^\infty(\mathbb{Z}, 1/v)$ and $c_0(\mathbb{Z}, 1/v)$.

Corollary 3.4. *Let v be a typical weight sequence on $\mathbb{Z}$ and let ψ be a function from $\mathbb{Z}$ into $\mathbb{C}$. Then the following statement are equivalent:*

i) $M_\psi \colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/v)$ *is compact.*

ii) $M_\psi \colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/v)$ *is compact.*

iii) $\lim_{|n|\to\infty} \psi(n) = 0$.

Corollary 3.5. *Let v be a typical weight sequence on $\mathbb{Z}$ and let φ be a function from $\mathbb{Z}$ into $\mathbb{Z}$. Then the following statement are equivalent:*

i) $C_\varphi \colon \ell^\infty(\mathbb{Z}, 1/v) \to \ell^\infty(\mathbb{Z}, 1/v)$ *is compact.*

ii) $C_\varphi \colon c_0(\mathbb{Z}, 1/v) \to c_0(\mathbb{Z}, 1/v)$ *is compact.*

iii) $\lim_{|n|\to\infty} (v(\varphi(n))/v(n)) = 0$.

If M_ψ and C_φ are bounded (compact), then $C_{\psi,\varphi} = M_\psi C_\varphi$ is bounded (compact). This is not the only situation for which $C_{\psi,\varphi}$ is bounded (compact). We give some examples.

Example 3.6. *For each $\alpha \geq 0$ consider the typical weight sequence on $\mathbb{Z}$ given by*

$$v_\alpha(n) = (1 + |n|)^\alpha \qquad (n \in \mathbb{Z}).$$

1. *Let $\psi = v_\beta$ with $0 < \beta < \alpha$ and $\varphi = 1_\mathbb{Z}$. Then M_ψ is not bounded on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$, C_φ is compact on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$, but $C_{\psi,\varphi}$ is compact on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$.*
2. *Let $\psi = e_1$ where $e_1(1) = 1$ and $e_1(n) = 0$ if $n \in \mathbb{Z} \setminus \{1\}$, and $\varphi(n) = n^2$ for all $n \in \mathbb{Z}$. Then M_ψ is compact on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$, C_φ is not bounded on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$, but $C_{\psi,\varphi}$ is compact on $\ell^\infty(\mathbb{Z}, 1/v_\alpha)$ and $c_0(\mathbb{Z}, 1/v_\alpha)$.*
3. *Let $\psi = 1/v_1$ and $\varphi(n) = [\sqrt{|n|}]$, where $[\cdot]$ is the integer part function given by*

   $$[x] = \max\{p \in \mathbb{Z} \colon p \leq x\} \qquad (x \in \mathbb{R}).$$

   Note that $[x] \leq x < [x] + 1$ for all $x \in \mathbb{R}$. Then neither ψ induces a compact multiplication operator from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$ nor φ induces a compact composition operator from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$, but $C_{\psi,\varphi}$ is compact from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$. The same assertion holds if these operators are defined from $c_0(\mathbb{Z}, 1/v_2)$ into $c_0(\mathbb{Z}, 1/v_1)$.
4. *Let $\psi = 1/v_{1/2}$ and $\varphi(n) = \lceil |n|^{3/4} \rceil$, where $\lceil \cdot \rceil$ is the ceiling function given by*

   $$\lceil x \rceil = \min\{p \in \mathbb{Z} \colon x \leq p\} \qquad (x \in \mathbb{R}).$$

   Note now that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ for all $x \in \mathbb{R}$. Then neither ψ induces a bounded multiplication operator from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$ nor φ induces a bounded composition operator from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$, but $C_{\psi,\varphi}$ is bounded from $\ell^\infty(\mathbb{Z}, 1/v_2)$ into $\ell^\infty(\mathbb{Z}, 1/v_1)$. This is also true if we define these operators from $c_0(\mathbb{Z}, 1/v_2)$ to $c_0(\mathbb{Z}, 1/v_1)$.

4. Weak compactness of $C_{\psi,\varphi}$

Finally, we show that every weakly compact operator $C_{\psi,\varphi}$ on $c_0(\mathbb{Z},1/v)$-spaces is compact, and deduce from this the same property for the weakly compact operators $C_{\psi,\varphi}$ on $\ell^\infty(\mathbb{Z},1/v)$-spaces.

Theorem 4.1. *Let v,w be weight sequences on $\mathbb{Z}$ such that v is typical and let ψ,φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z})\subset\mathbb{Z}$. Then every weakly compact operator $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is compact.*

Proof. By Gantmacher's Theorem, the operator $C_{\psi,\varphi}$ is weakly compact from $c_0(\mathbb{Z},1/v)$ to $c_0(\mathbb{Z},1/w)$ if and only if $C^{**}_{\psi,\varphi}(c_0(\mathbb{Z},1/v)^{**})\subset J(c_0(\mathbb{Z},1/w))$, where J denotes the canonical injection of $c_0(\mathbb{Z},1/w)$ into $c_0(\mathbb{Z},1/w)^{**}$.

We now see that

$$\begin{aligned}C^{**}_{\psi,\varphi}(c_0(\mathbb{Z},1/v)^{**})\subset J(c_0(\mathbb{Z},1/w)) &\Leftrightarrow \Phi C^{**}_{\psi,\varphi}\Phi^{-1}(\ell^\infty(\mathbb{Z},1/v))\subset c_0(\mathbb{Z},1/w)\\ &\Leftrightarrow \widetilde{C}_{\psi,\varphi}(\ell^\infty(\mathbb{Z},1/v))\subset c_0(\mathbb{Z},1/w),\end{aligned}$$

where $\widetilde{C}_{\psi,\varphi}$ denotes the weighted composition operator on $\ell^\infty(\mathbb{Z},1/v)$ with weight ψ and symbol φ. We have used the equation (5) in the first equivalence, and Lemma 2.3 in the second one.

Therefore $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is weakly compact if and only if $\psi\cdot(\lambda\circ\varphi)\in c_0(\mathbb{Z},1/w)$ for all $\lambda\in\ell^\infty(\mathbb{Z},1/v)$.

Assume that $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is weakly compact. Since $v\in\ell^\infty(\mathbb{Z},1/v)$, it follows that $\psi\cdot(v\circ\varphi)\in c_0(\mathbb{Z},1/w)$, and then $C_{\psi,\varphi}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is compact by Theorem 3.2. $\square$

Corollary 4.2. *Let v,w be weight sequences on $\mathbb{Z}$ such that v is typical and let ψ,φ be complex-valued functions on $\mathbb{Z}$ such that $\varphi(\mathbb{Z})\subset\mathbb{Z}$. Then every weakly compact operator $C_{\psi,\varphi}\colon \ell^\infty(\mathbb{Z},1/v)\to\ell^\infty(\mathbb{Z},1/w)$ is compact.*

Proof. Assume $C_{\psi,\varphi}\colon\ell^\infty(\mathbb{Z},1/v)\to\ell^\infty(\mathbb{Z},1/w)$ is weakly compact. Since

$$\Phi\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}\Phi^{-1}=C_{\psi,\varphi}$$

by Lemma 2.3, it follows that the operator $\left(C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\right)^{**}$ is weakly compact from $c_0(\mathbb{Z},1/v)^{**}$ to $c_0(\mathbb{Z},1/w)^{**}$ and so also is the operator $C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}$ from $c_0(\mathbb{Z},1/v)$ to $c_0(\mathbb{Z},1/w)$.

Then, according to Theorem 4.1, $C_{\psi,\varphi}|_{c_0(\mathbb{Z},1/v)}\colon c_0(\mathbb{Z},1/v)\to c_0(\mathbb{Z},1/w)$ is compact. Thus, by applying Theorem 3.2 and Corollary 3.3, we conclude that $C_{\psi,\varphi}\colon\ell^\infty(\mathbb{Z},1/v)\to\ell^\infty(\mathbb{Z},1/w)$ is compact. $\square$

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